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On the decay of low-magnetic-Reynolds-number turbulence in an imposed magnetic field

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We examine the integral properties of freely decaying homogeneous magnetohydrodynamic (MHD) turbulence subject to an imposed magnetic field $\boldsymbol{B}_0$ at low-magnetic Reynolds number. We confirm that, like conventional isotropic turbulence, the fully developed state possesses a Loitsyansky-like integral invariant, in this case $I_{//} = -\int r_\perp^2 \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle \, \mathrm{d}\boldsymbol{r}$, where $\langle \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}\boldsymbol{c}) \rangle = \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ is the usual two-point velocity correlation and the subscript $\perp$ indicates components perpendicular to the imposed field. The conservation of $I_{//}$ for fully developed turbulence places a fundamental restriction on the way in which the integral scales can develop, i.e. it implies $u_\perp^2 \ell_\perp^4 \ell_{//} \approx$ constant where $u_\perp$, $\ell_\perp$ and $\ell_{//}$ are integral scales. This constraint can be used to estimate the evolution of $u_\perp(t;\boldsymbol{B}_0)$, $\ell_\perp(t;\boldsymbol{B}_0)$ and $\ell_{//}(t;\boldsymbol{B}_0)$, and these theoretical decay laws are shown to be in good agreement with numerical simulations.

1. Introduction

1.1. *The governing equations*

We are interested in the turbulent motion of a conducting fluid which evolves in the presence of an imposed magnetic field $\boldsymbol{B}_0$. Such flows are important in the metallurgical industry, where magnetic fields are routinely used to control the motion of liquid metals, in geodynamo theory, and in astrophysics (Davidson 1999, 2004). For simplicity we shall take the fluid to be incompressible and Newtonian, so the flow is governed by the Navier–Stokes equation and subject to the Lorentz force $\boldsymbol{j} \times \boldsymbol{B}$, where $\boldsymbol{j}$ is the current density and $\boldsymbol{B}$ is the total magnetic field, $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$. Here $\boldsymbol{b}$ is the field associated with the induced currents through Ampère's law, $\nabla \times \boldsymbol{b} = \mu \boldsymbol{j}$, where μ is the permeability of free space.

It is conventional to characterize the behaviour of magnetohydrodynamic (MHD) turbulence using two dimensionless groups:

$$R_m = \frac{u\ell}{\lambda} = \sigma\mu u\ell, \tag{1.1}$$

$$N = \frac{\sigma B_0^2 \ell}{\rho u} = \frac{\ell/u}{\tau}, \tag{1.2}$$

where u and ℓ are the integral scales of the turbulent motion, ρ the fluid density, σ the electrical conductivity, $\lambda = (\sigma\mu)^{-1}$ the magnetic diffusivity and $\tau = (\sigma B_0^2/\rho)^{-1}$ the

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Joule damping time. The so-called interaction parameter N provides one measure of the relative strengths of the Lorentz and inertial forces, while the magnetic Reynolds number R_m measures the relative importance of the diffusion and convection of magnetic field lines (see e.g. Roberts 1967).

When R_m is small, i.e. the fluid is a relatively poor conductor, the currents induced by motion across the imposed magnetic field are also small. In such a situation, it is readily confirmed that $|\boldsymbol{b}| \ll |\boldsymbol{B}_0|$, while Ohm's law, $\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$, simplifies to

$$\boldsymbol{j} = \sigma(-\nabla\psi + \boldsymbol{u} \times \boldsymbol{B}_0), \tag{1.3}$$

where $\boldsymbol{E}$ is the electric field and ψ the electrostatic potential. Evidently, $\boldsymbol{j}$ is now uniquely determined by the two expressions $\nabla \cdot \boldsymbol{j} = 0$ and $\nabla \times \boldsymbol{j} = \sigma(\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{u}$. Virtually all terrestrial MHD satisfies $R_m \ll 1$, with the exception of the geodynamo, and this is the regime which is of particular interest in this paper. When R_m is small the solenoidal part of the Lorentz force per unit mass $\boldsymbol{F}$ is a linear functional of $\boldsymbol{u}$, according to

$$\nabla^2 \boldsymbol{F} = -\frac{1}{\rho}\nabla \times \nabla \times (\boldsymbol{j} \times \boldsymbol{B}_0) = -\frac{1}{\rho}\nabla \times (\boldsymbol{B}_0 \cdot \nabla \boldsymbol{j}) = -\frac{\sigma}{\rho}(\boldsymbol{B}_0 \cdot \nabla)^2 \boldsymbol{u},$$

which we can rewrite as

$$\boldsymbol{F} = -\frac{\sigma}{\rho}\nabla^{-2}(\boldsymbol{B}_0 \cdot \nabla)^{-2}\boldsymbol{u}, \tag{1.4}$$

where ∇^{-2} is a symbolic operator defined via the Biot–Savart law. Our governing equations are then

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla(p/\rho) - \frac{\sigma}{\rho}\nabla^{-2}(\boldsymbol{B}_0 \cdot \nabla)^2 \boldsymbol{u} + \nu \nabla^2 \boldsymbol{u}, \nabla \cdot \boldsymbol{u} = 0.$$

We are interested in freely decaying statistically axisymmetric MHD turbulence and our aim is to characterize the behaviour of the large scales. It is well known that such turbulence becomes anisotropic as the large eddies elongate in the direction of the imposed magnetic field through Alfven wave propagation. So it is natural to ask how u^2, $\ell_{//}$ and $\ell_{\perp}$ evolve as a function of time, and how this evolution depends on the magnitude of $\boldsymbol{B}_0$, and hence on N. Here the subscripts // and $\perp$ indicate directions parallel and perpendicular to the imposed field $\boldsymbol{B}_0$, with $\ell_{//}$ and $\ell_{\perp}$ defined by

$$\ell_{//} = \left\langle u_{//}^2 \right\rangle^{-1} \int_0^{\infty} \langle u_{//}(\boldsymbol{x}) u_{//}(\boldsymbol{x} + r\hat{\boldsymbol{e}}_{//}) \rangle \, \mathrm{d}r,$$

$$\ell_{\perp} = \left\langle u_{\perp}^2 \right\rangle^{-1} \int_0^{\infty} \langle u_{\perp}(\boldsymbol{x}) u_{\perp}(\boldsymbol{x} + r\hat{\boldsymbol{e}}_{\perp}) \rangle \, \mathrm{d}r,$$

where $\hat{\boldsymbol{e}}_{//}$ and $\hat{\boldsymbol{e}}_{\perp}$ are unit vectors and $u_{\perp}$ is any component of $\boldsymbol{u}$ which is perpendicular to $\boldsymbol{B}_0$.

We shall propose explicit predictions for $u^2(N;t)$, $\ell_{//}(N;t)$ and $\ell_{\perp}(N;t)$ and then test these predictions against direct numerical simulations (DNS) of the Navier–Stokes equation, modified to incorporate the Lorentz force, performed in large computational domains. A recurring theme throughout the paper is the essential role played by angular momentum conservation, as this provides the key to determining the behaviour of u^2, $\ell_{//}$ and $\ell_{\perp}$.

1.2. *The role of angular momentum conservation in homogeneous MHD turbulence*

The importance of angular momentum conservation for homogeneous MHD turbulence was first pointed out by Davidson (1997). It is useful to summarize briefly the arguments here, partly to set the scene for the rest of the paper, and partly in order to emphasize the uncertainties inherent in the earlier analysis. Following Davidson (1997), we start with inhomogeneous, rather than with homogeneous, turbulence, and with a thought experiment.

Suppose our conducting fluid is held in a large, insulated, spherical domain of radius R. A uniform field $\boldsymbol{B}_0$ is imposed on the fluid which, at $t=0$, is set into turbulent motion. Although we are primarily interested in the case of low R_m, we shall start by taking both N and R_m to be of arbitrary size. The Reynolds number, on the other hand, is taken to be large, $Re = u\ell/\nu \gg 1$, and the flow domain to be big, in the sense that $R \gg \ell$.

For $t > 0$ the energy of the turbulence (kinetic energy plus magnetic energy) decays, partly through viscous stresses and partly through Joule dissipation, $\boldsymbol{j}^2/\sigma$. However, this decay is subject to a powerful constraint: that of angular momentum conservation. Let $\boldsymbol{H}$ be the global angular momentum of the fluid, $\boldsymbol{H} = \int_V \boldsymbol{x} \times \boldsymbol{u}\,\mathrm{d}V$, and $\boldsymbol{T}$ the net torque associated with the Lorentz force,

$$\boldsymbol{T} = \int_V \boldsymbol{x} \times (\boldsymbol{j} \times \boldsymbol{b})\,\mathrm{d}V + \int_V \boldsymbol{x} \times (\boldsymbol{j} \times \boldsymbol{B}_0)\,\mathrm{d}V.$$

Now a closed system of currents cannot produce any net torque when it interacts with its self-field $\boldsymbol{b}$ and so the first integral is zero. The second integral can be rewritten using the identity

$$2\boldsymbol{x} \times (\boldsymbol{j} \times \boldsymbol{B}_0) = (\boldsymbol{x} \times \boldsymbol{j}) \times \boldsymbol{B}_0 + \boldsymbol{j} \cdot \nabla(\boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{B}_0)), \tag{1.5}$$

the fact that $\boldsymbol{j}$ is solenoidal, and that $\boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S} = 0$, to yield

$$\boldsymbol{T} = \frac{1}{2} \int_V (\boldsymbol{x} \times \boldsymbol{j})\,\mathrm{d}V \times \boldsymbol{B}_0 = \boldsymbol{m} \times \boldsymbol{B}_0, \tag{1.6}$$

where $\boldsymbol{m}$ is the net dipole moment associated with $\boldsymbol{j}$. It follows that, if we neglect the viscous torque on the outer boundary, which is valid for times much greater than ℓ/u when $R \gg \ell$, then the component of $\boldsymbol{H}$ parallel to $\boldsymbol{B}_0$ is conserved, $\boldsymbol{H}_{/\!/} = \text{constant}$. Moreover, it can be shown (Davidson 2004, pp. 533–534) that the continual destruction of energy, subject to the constraint of $\boldsymbol{H}_{/\!/} = \text{constant}$, inevitably leads to a two-dimensionalization of the flow orientated by the mean field, $\boldsymbol{B}_0$. This prediction holds for any value of R_m and is consistent with energy spreading along the mean field lines through Alfven wave propagation.

The behaviour of the components of $\boldsymbol{H}$ normal to $\boldsymbol{B}_0$ can also be predicted, at least for the particular case of low-R_m turbulence. That is, (1.3) can be combined with (1.5), but with $\boldsymbol{j}$ in (1.5) replaced by $\boldsymbol{u}$, to evaluate the dipole moment defined by (1.6):

$$\boldsymbol{m} = \frac{\sigma}{2} \int \nabla \times (\psi \boldsymbol{x})\,\mathrm{d}V + \frac{\sigma}{2} \int \boldsymbol{x} \times (\boldsymbol{u} \times \boldsymbol{B}_0)\,\mathrm{d}V = \frac{\sigma}{4} \int (\boldsymbol{x} \times \boldsymbol{u})\,\mathrm{d}V \times \boldsymbol{B}_0.$$

(The integral involving ψ converts to a surface integral which is zero for any spherical surface.) Evidently $\boldsymbol{m} = (1/4)\sigma \boldsymbol{H} \times \boldsymbol{B}_0$, from which $\boldsymbol{T} = -(1/4)\sigma B_0^2 \boldsymbol{H}_\perp$. It follows that, since there is no net torque associated with either pressure or inertial forces,

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = -\frac{\boldsymbol{H}_\perp}{4\tau}, \tag{1.7}$$

and hence

$$\boldsymbol{H}_{/\!/} = \text{constant} \quad (\text{any } N,\ \text{any } R_m) \tag{1.8}$$

and (Davidson 1995)

$$\boldsymbol{H}_\perp = \boldsymbol{H}_\perp(0)\exp(-t/4\tau) \quad (\text{any } N, \text{low } R_m). \tag{1.9}$$

Note that the angular momentum $\boldsymbol{H}$ does not depend on the choice of origin as the integral of $\boldsymbol{u}$ is zero for a closed domain.

So far we have considered inhomogeneous turbulence. However, it turns out that expression (1.8) is also potentially important for homogeneous MHD turbulence. To see why, we must consider Landau's analysis of conventional, isotropic turbulence ($\boldsymbol{B}=0$).

Landau showed that the net angular momentum of a cloud of turbulence evolving in a large, closed sphere ($R \gg \ell$) can be related to the two-point velocity correlation, $\langle \boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r})\rangle = \langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle$, as follows. Noting that $\boldsymbol{u}\cdot \mathrm{d}\boldsymbol{S}=0$ on $|\boldsymbol{x}|=R$, we have

$$\boldsymbol{H}^2 = \int_V \boldsymbol{x}\times\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\cdot\int_V \boldsymbol{x}'\times\boldsymbol{u}'\,\mathrm{d}\boldsymbol{x}' = \int\int(\boldsymbol{x}'-\boldsymbol{x})^2\boldsymbol{u}\cdot\boldsymbol{u}'\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{x}, \tag{1.10}$$

and on ensemble averaging this yields

$$\langle \boldsymbol{H}^2\rangle = \int_V\left[-\int_{V^*}\boldsymbol{r}^2\langle\boldsymbol{u}\cdot\boldsymbol{u}'\rangle\,\mathrm{d}\boldsymbol{r}\right]\mathrm{d}\boldsymbol{x}, \tag{1.11}$$

where $\boldsymbol{r}=\boldsymbol{x}'-\boldsymbol{x}$ and $\boldsymbol{u}'=\boldsymbol{u}(x')$. Note that the shape of V^* depends on the location of $\boldsymbol{x}$ within V. Landau then made the crucial assumption that all two-point correlations, $\langle u_i u_j'\rangle$, fall off rapidly (say exponentially) with separation $|\boldsymbol{r}|$, and on a scale of ℓ, so that only the thin layer of turbulence close to the surface $|\boldsymbol{x}|=R$ is aware of that boundary. In such a case we might expect that the bulk of the turbulence will behave as if it were isotropic. Moreover, we might ignore the contribution to $\langle\boldsymbol{H}^2\rangle$ which comes from the thin layer of anisotropic turbulence adjacent to $|\boldsymbol{x}|=R$. We can then treat $\langle\boldsymbol{u}\cdot\boldsymbol{u}'\rangle$ in (1.11) as isotropic. We can also ignore far-field contributions to the inner integral in (1.11), which allows us replace V^* by V_∞, indicating an integration over all $\boldsymbol{r}$. If all of this is true, then we have

$$\langle \boldsymbol{H}^2\rangle \approx \int_V\left[-\int_{V_\infty}\boldsymbol{r}^2\langle\boldsymbol{u}\cdot\boldsymbol{u}'\rangle\,\mathrm{d}\boldsymbol{r}\right]\mathrm{d}\boldsymbol{x}, \quad \ell \ll R, \tag{1.12}$$

and in the limit of $R/\ell\to\infty$, this yields (Landau & Lifshitz 1959)

$$\langle \boldsymbol{H}^2\rangle/V = -\int_{V_\infty}\boldsymbol{r}^2\langle\boldsymbol{u}\cdot\boldsymbol{u}'\rangle\,\mathrm{d}\boldsymbol{r}, \tag{1.13}$$

The integral on the right-hand side is known as Loitsyansky's integral, which we shall denote by I.

So far we have discussed only kinematics. We now turn to dynamics. If, once again, we ignore the viscous torque on the outer boundary, we have

$$\langle \boldsymbol{H}^2\rangle/V = -\int_{V_\infty}\boldsymbol{r}^2\langle\boldsymbol{u}\cdot\boldsymbol{u}'\rangle\,\mathrm{d}\boldsymbol{r} = I = \text{constant} \quad (\text{isotropic turbulence}), \tag{1.14}$$

a result which should apply to homogeneous isotropic turbulence.

Actually, as pointed out by Monin & Yaglom (1975), this derivation of

$$I = -\int_{V_\infty}\boldsymbol{r}^2\langle\boldsymbol{u}\cdot\boldsymbol{u}'\rangle\,\mathrm{d}\boldsymbol{r} = \text{constant} \tag{1.15}$$

is not rigorous because it is not at all clear that it is legitimate to ignore the contribution to $\langle \boldsymbol{H}^2 \rangle$ that comes from the thin layer of fluid adjacent to the boundary. This is the case even if $\langle u_i u'_j \rangle_\infty$ falls off rapidly with $|\boldsymbol{r}|$. (Here the subscript ∞ indicates $|\boldsymbol{r}| \to \infty$.) However, it turns out to be possible to use angular momentum conservation in a slightly different way to establish (1.15) in a rigorous fashion, assuming of course that there is a rapid fall-off of $\langle u_i u'_j \rangle_\infty$ (Davidson 2009). In addition, under the same assumption about $\langle u_i u'_j \rangle_\infty$, which in turn requires the triple correlations to decay rapidly, it is possible to deduce (1.15) directly from the Kármán–Howarth equation.

Let us now apply the same logic to MHD turbulence. From (1.8) we might expect that (Davidson 1997)

$$\langle \boldsymbol{H}_{//}^2 \rangle / V = -\int_{V_\infty} \boldsymbol{r}_\perp^2 \, \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle \, \mathrm{d}\boldsymbol{r} = I_{//} = \text{constant} \tag{1.16}$$

(axisymmetric turbulence, any N, any R_m).

Note that, as emphasized in Davidson (2004, p. 542), (1.16) applies equally to low- and high-R_m turbulence, and so is relevant to both terrestrial and astrophysical MHD, and indeed there is some support for (1.16) in the high-R_m simulations of Bigot, Galtier & Politano (2008). The would-be invariant of (1.16) also holds in rotating and stratified turbulence (Davidson 2004, 2009). However, as for its isotropic counterpart, (1.16) is not rigorous because it pre-supposes that the thin layer of fluid near the boundary makes no significant contribution to $\langle \boldsymbol{H}^2 \rangle$ in the limit of $R/\ell \to \infty$. Worse, it can be valid only if the two-point double correlations, $\langle u_i u'_j \rangle$, fall off rapidly enough with separation $|\boldsymbol{r}|$, and it is by no means obvious that this should be the case. Indeed, ever since Batchelor & Proudman (1956), it is often argued that, in isotropic turbulence, $\langle u_i u'_j \rangle_\infty$ and $\langle u_i u_j u'_k \rangle_\infty$ fall off as power laws: $\langle u_i u'_j \rangle_\infty \sim r^{-6}$ and $\langle u_i u_j u'_k \rangle_\infty \sim r^{-4}$. If this is indeed the case, then the Kármán–Howarth equation tells us that I in (1.15) is not conserved (see (1.36)). Since (1.15) is in doubt, so too is (1.16).

All in all, it would seem that the validity or otherwise of (1.16) is far from clear-cut. Consequently, we shall adopt the position that (1.16) is merely suggestive, its validity to be assessed through a comparison with the results of the DNS.

1.3. *Decay laws for MHD turbulence at low R_m*

Kolmogorov (1941) used the alleged invariance of I in isotropic turbulence to predict the rate of decay of energy. Let us define u via the expression $u^2 = (1/3)\langle \boldsymbol{u}^2 \rangle$. Since the large scales in isotropic turbulence are self-similar when scaled on u and ℓ, (1.15), which is dominated by the large scales, demands

$$u^2 \ell^5 = \text{constant}. \tag{1.17}$$

This may be combined with the empirical (but well-established) law

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} = -\alpha \frac{u^3}{\ell}, \quad \alpha = \text{constant}, \tag{1.18}$$

to give

$$\frac{u^2}{u_0^2} = \left[1 + \frac{7\alpha}{10}\left(\frac{u_0 t}{\ell_0}\right)\right]^{-10^{-7}}, \tag{1.19}$$

$$\frac{\ell}{\ell_0} = \left[1 + \frac{7\alpha}{10}\left(\frac{u_0 t}{\ell_0}\right)\right]^{2/7}, \tag{1.20}$$

where the dimensionless constant α is of the order of unity and u_0 and ℓ_0 are the initial values of u and ℓ. Although the alleged invariance of I has been hotly disputed, these decay laws, as well as the prediction that I is constant, are well supported by DNS in large computational domains (Ishida, Davidson & Kaneda 2006), provided the turbulence is allowed some time to become fully developed.

The analogous decay laws for homogeneous low-R_m MHD turbulence are set out in Davidson (2001, p. 253), and in more detail in Davidson (2004, pp. 543–544). Let us make the assumption that (1.16) is indeed valid. Then the analogue of (1.17) is, from (1.16),

$$u^2\ell_\perp^4\ell_{//} = \text{constant}, \tag{1.21}$$

and the corresponding energy equation is

$$\frac{\partial}{\partial t}\frac{1}{2}\left\langle \boldsymbol{u}^2\right\rangle = -\nu\left\langle \boldsymbol{\omega}^2\right\rangle - \left\langle \boldsymbol{j}^2\right\rangle/\rho\sigma, \quad \boldsymbol{\omega} = \nabla\times\boldsymbol{u}. \tag{1.22}$$

(Note that the magnetic energy density, $\langle \boldsymbol{b}^2/2\rho\mu\rangle$, is of order $NR_m\langle\boldsymbol{u}^2\rangle$, and so may be neglected in the low-R_m approximation.) The viscous dissipation is now modelled in the usual way, as $\nu\langle\boldsymbol{\omega}^2\rangle \sim \alpha u^3/\ell_\perp$, while the magnitude of the Ohmic dissipation can be estimated from $\nabla\times\boldsymbol{j} = \sigma(\boldsymbol{B}_0\cdot\nabla)\boldsymbol{u}$, which yields

$$\frac{\left\langle \boldsymbol{j}^2\right\rangle}{\rho\sigma} \sim \left(\frac{\ell_\perp}{\ell_{//}}\right)^2\frac{\left\langle \boldsymbol{u}^2\right\rangle}{\tau} = \frac{\beta}{2}\left(\frac{\ell_\perp}{\ell_{//}}\right)^2\frac{\left\langle \boldsymbol{u}^2\right\rangle}{\tau}, \tag{1.23}$$

for some coefficient β which is of the order of unity. (It can be shown that $\beta = 2/3$ for isotropic turbulence.) Thus, in Davidson (2001, 2004) the energy equation is modelled as

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} = -\alpha\frac{u^3}{\ell_\perp} - \beta\left(\frac{\ell_\perp}{\ell_{//}}\right)^2\frac{u^2}{\tau}, \tag{1.24}$$

where, as before, $u^2 = (1/3)\langle\boldsymbol{u}^2\rangle$, and the coefficients α and β, defined by

$$\alpha = \frac{2\nu\left\langle \boldsymbol{\omega}^2\right\rangle}{3u^3/\ell_\perp} \tag{1.25}$$

and

$$\beta = \frac{2\langle \boldsymbol{j}^2\rangle/\rho\sigma}{(\ell_\perp/\ell_{//})^2\langle\boldsymbol{u}^2\rangle/\tau}, \tag{1.26}$$

are treated as constants of order unity. Note that the ratio of Ohmic to viscous dissipation is of order N. Note also that the modelling of the viscous term in (1.24) is unlikely to be accurate when N is large and the flow strongly anisotropic. However, the viscous dissipation is relatively unimportant in such cases, so this does not present a problem.

In the limit of $N = 0$ we recover (1.19) and (1.20) from (1.21) and (1.24). For $N\to\infty$, on the other hand, inertia is negligible and our system reduces to

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} = -\beta\left(\frac{\ell_\perp}{\ell_{//}}\right)^2\frac{u^2}{\tau}, \quad u^2\ell_\perp^4\ell_{//} = \text{constant}. \tag{1.27}$$

It is well known that $\ell_\perp$ remains essentially constant during the decay of high-N turbulence, and so (1.27) integrates to give

$$\frac{u^2}{u_0^2} = \left[1 + \frac{2\beta t}{\tau}\right]^{-1/2}, \quad \frac{\ell_{//}}{\ell_0} = \left[1 + \frac{2\beta t}{\tau}\right]^{1/2}, \tag{1.28}$$

scalings which may also be obtained by direct integration of the linear equation (Moffatt 1967)

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\left(\frac{p}{\rho}\right) - \frac{1}{\tau}\nabla^{-2}\frac{\partial^2}{\partial x_{//}^2}\boldsymbol{u}. \tag{1.29}$$

For intermediate values of N, however, there is a problem as (1.21) and (1.24) have, between them, three unknowns: u^2, $\ell_{//}$ and $\ell_\perp$. To close the system Davidson (2001) proposed using

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\ell_{//}}{\ell_\perp}\right)^2 = \frac{2\beta}{\tau}, \tag{1.30}$$

which is exact for $N \to 0$ and $N \to \infty$, and may be thought of as an interpolation formula for intermediate N. If we adopt (1.30), then (1.24) integrates to give

$$\frac{u^2}{u_0^2} = \left[1 + \frac{7\alpha}{15\beta}\frac{\left(\hat{t}^{3/4}-1\right)}{N_0}\right]^{-10/7}\hat{t}^{-1/2}, \tag{1.31}$$

$$\frac{\ell_\perp}{\ell_0} = \left[1 + \frac{7\alpha}{15\beta}\frac{\left(\hat{t}^{3/4}-1\right)}{N_0}\right]^{2/7}, \tag{1.32}$$

$$\frac{\ell_{//}}{\ell_0} = \left[1 + \frac{7\alpha}{15\beta}\frac{\left(\hat{t}^{3/4}-1\right)}{N_0}\right]^{2/7}\hat{t}^{1/2}, \tag{1.33}$$

where N_0 is the initial value of N and $\hat{t} = 1 + 2\beta t/\tau$ (Davidson 2004). It is readily confirmed that (1.19), (1.20) and (1.28) may be recovered from (1.31)–(1.33) in the appropriate limits. Moreover, for the particular case of $N_0 = 7\alpha/15\beta$, we obtain the simple power laws

$$u^2 \sim u_0^2\hat{t}^{-11/7}, \quad \ell_{//} \sim \ell_0\hat{t}^{5/7}, \quad \ell_\perp \sim \ell_0\hat{t}^{3/14}. \tag{1.34}$$

1.4. *Uncertainties and questions*

There are at least four unjustified approximations inherent in (1.31)–(1.33). First, the model treats α and β as constant during the decay. In practice, however, these coefficients could be sensitive to the degree of anisotropy of the turbulence and, since this grows with time, we might anticipate that α and β will vary throughout the decay. Second, there is no formal justification for the heuristic equation (1.30), except that it holds for small and large N. Would we really expect

$$C = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\ell_{//}}{\ell_\perp}\right)^2 \bigg/ \frac{2\beta}{\tau} \tag{1.35}$$

to be constant and equal to unity for all N_0 and all t? We might classify these first two approximations as issues of modelling. The third problem, however, is more fundamental. The move from the inhomogeneous prediction (1.8) to the homogeneous expression (1.16), which mimics Landau's analysis of isotropic turbulence, rests crucially on the assumption that $\langle u_i u_j' \rangle(\boldsymbol{r})$ decays rapidly (say exponentially) with separation $|\boldsymbol{r}|$. If this is not the case then (1.16) is suspect, (1.21) cannot be defended, and predictions (1.31)–(1.33) must be abandoned. The fourth (and final) uncertainty relates to self-similarity. To establish (1.21) from (1.16) we require that the large scales are self-similar, at least in some approximate sense. While there is ample evidence that this is the case in conventional hydrodynamic turbulence, it is less certain that this is so in MHD, where the evidence is more scant.

One of the purposes of this paper is to probe the validity of these four approximations, and hence test the accuracy of predictions (1.31)–(1.33). We shall return to the issues of whether or not α, β and C are really constants, and whether the large scales are self-similar, when we examine the numerical data in § 3. In this section we focus on the more fundamental issue of whether or not $\langle u_i u_j' \rangle(\boldsymbol{r})$ is likely to decay sufficiently rapidly with $|\boldsymbol{r}|$ for (1.16) to be a good approximation. To this end we first consider the simpler case of conventional isotropic turbulence ($\boldsymbol{B}=0$), which is well documented.

Here integration of the Kármán–Howarth equation yields

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 8\pi \left[u^3 r^4 K(r)\right]_\infty, \tag{1.36}$$

where $u^3 K(r) = \langle u_x^2(\boldsymbol{x}) u_x(\boldsymbol{x} + r\hat{\boldsymbol{e}}_x)\rangle$ is the usual longitudinal correlation function (see e.g. Davidson 2004). So the fate of I depends crucially on the behaviour of $\langle u_x^2 u_x' \rangle_\infty$. Now Batchelor & Proudman (1956) showed that the long-range pressure forces in homogeneous turbulence are capable of establishing long-range pressure–velocity correlations of the form $\langle u_i u_j p' \rangle_\infty \sim r^{-3}$. That is, a fluctuation in $\boldsymbol{u}$ at a point $\boldsymbol{x}$ sets up pressure waves (which travel infinitely fast in an incompressible fluid) and inverting the Poisson equation

$$\nabla^2 p = -\rho \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} \tag{1.37}$$

shows that the resulting pressure fluctuations fall off from the source as $p' \sim r^{-3}$, leading to $\langle u_i u_j p' \rangle_\infty \sim r^{-3}$. Since the triple correlations are governed by an equation of the form

$$\rho \frac{\partial}{\partial t} \langle u_i u_j u_k' \rangle = -\frac{\partial}{\partial r_k} \langle u_i u_j p' \rangle + \cdots, \tag{1.38}$$

we might expect $\langle u_i u_j u_k' \rangle_\infty$ to fall as $\langle u_i u_j u_k' \rangle_\infty = c_{ijk} r^{-4}$. The Kármán–Howarth equation then demands that $\langle u_i u_j' \rangle_\infty \sim d_{ij} r^{-6}$, where c_{ijk} and d_{ij} are dimensional pre-factors. This is enough to suggest that the right-hand side of (1.36) is non-zero and I time-dependent, in direct contravention of (1.15).

In summary, then, the non-local nature of the pressure equation, $p/\rho = -\nabla^{-2}[\partial^2 u_i u_j / \partial x_i \partial x_j]$, is enough to set up long-range triple correlations, $\langle u_i u_j u_k' \rangle_\infty = c_{ijk} r^{-4}$, whose strengths are, in principle, sufficient to make I time-dependent. The same triple correlations also induce long-range double correlations, $\langle u_i u_j' \rangle_\infty \sim d_{ij} r^{-6}$, which call into question the validity of Landau's derivation of (1.15). However, it is important to keep in mind that there is no rigorous theory which can predict the magnitude of the pre-factors c_{ijk} and d_{ij}. Moreover, the numerical simulations of Ishida *et al.* (2006) show that, once the turbulence is fully developed, these pre-factors are extremely small, so that, to within a reasonable degree of accuracy, one recovers the classical predictions of $I = \text{constant}$ and $u^2 \sim t^{-10/7}$. (During an initial transient, I is observed to be time-dependent in these simulations, but this probably an artifact of the somewhat artificial initial conditions used in the computations, i.e. random phases in the Fourier modes.)

Let us now return to the case of low-R_m MHD turbulence. As before, the non-local pressure force is capable of establishing triple correlations of the form $\langle u_i u_j u_k' \rangle_\infty = c_{ijk} r^{-4}$. This time, however, the turbulence is anisotropic and so the generalized Kármán–Howarth equation suggests $\langle u_i u_j' \rangle_\infty \sim r^{-5}$, rather than the r^{-6}

fall predicted for isotropic turbulence. (Symmetry kills off the r^{-5} term in isotropic turbulence.) However, the situation is more complicated in MHD turbulence, since $\boldsymbol{J}$ is also a non-local function of $\boldsymbol{u}$, i.e. $\nabla \times \boldsymbol{J} = \sigma\,(\boldsymbol{B}_0 \cdot \nabla)\,\boldsymbol{u}$, which yields a non-local Lorentz force,

$$\boldsymbol{F} = -\frac{\sigma}{\rho}\nabla^{-2}\left[(\boldsymbol{B}_0 \cdot \nabla)^2\,\boldsymbol{u}\right] = -\frac{1}{\tau}\nabla^{-2}\frac{\partial^2 \boldsymbol{u}}{\partial x_{//}^2}. \tag{1.39}$$

The governing equation for $\langle u_i u_j' \rangle(\boldsymbol{r}, t)$ is then

$$\frac{\partial}{\partial t}\left\langle u_i u_j' \right\rangle = \nabla \cdot \left[\langle uuu' \rangle , \langle pu' \rangle\right] - \frac{2}{\tau}\frac{\partial^2}{\partial r_{//}^2}\nabla^{-2}\left\langle u_i u_j' \right\rangle, \tag{1.40}$$

or equivalently,

$$\frac{\partial}{\partial t}\left\langle u_i u_j' \right\rangle = \nabla \cdot \left[\langle uuu' \rangle , \langle pu' \rangle\right] + \frac{1}{2\pi\tau}\frac{\partial^2}{\partial r_{//}^2}\int \frac{\left\langle u_i u_j' \right\rangle(r^*)}{|\boldsymbol{r} - \boldsymbol{r}^*|}\,\mathrm{d}\boldsymbol{r}^*. \tag{1.41}$$

Here $\nabla \cdot [\langle uuu' \rangle, \langle pu' \rangle]$ represents the conventional inertial and pressure terms in the Kármán–Howarth equation,

$$\nabla \cdot \left[\langle uuu' \rangle , \langle pu' \rangle\right] = \frac{\partial}{\partial r_k}\left[\left\langle u_i u_k u_j' \right\rangle - \left\langle u_j' u_k' u_i \right\rangle\right] + \frac{\partial}{\partial r_i}\left\langle p u_j' \right\rangle - \frac{\partial}{\partial r_j}\left\langle p' u_i \right\rangle. \tag{1.42}$$

Expanding the integral on the right-hand side of (1.41), and using $\nabla \cdot \boldsymbol{u} = 0$, leads to far-field terms of order r^{-5}, and so we have (Davidson 1997)

$$\frac{\partial}{\partial t}\left\langle u_i u_j' \right\rangle_\infty = \left[\nabla \cdot \left(\langle uuu' \rangle , \langle pu' \rangle\right)\right]_\infty + O(\tau^{-1} r^{-5}). \tag{1.43}$$

In summary, then, there are two potential sources of long-range double correlations, $\langle u_i u_j' \rangle_\infty$. On the one hand the non-local pressure fluctuations can lead to long-range triple correlations, $\langle u_i u_j u_k' \rangle_\infty = c_{ijk} r^{-4}$, and hence to $\langle u_i u_j' \rangle_\infty \sim d_{ij}^{(p)} r^{-5}$, where the time derivatives of the $d_{ij}^{(p)}$ are dependent on the c_{ijk}. On the other hand, the non-local dependence of $\boldsymbol{J}$ on $\boldsymbol{u}$ leads directly to $\langle u_i u_j' \rangle_\infty \sim d_{ij}^{(J)} r^{-5}$, an effect that bypasses the triple correlations. The combination of the two processes yields

$$\left\langle u_i u_j' \right\rangle_\infty \sim \left[d_{ij}^{(p)} + d_{ij}^{(J)}\right] r^{-5} = d_{ij} r^{-5}, \tag{1.44}$$

where $d_{ij} = d_{ij}^{(p)} + d_{ij}^{(J)}$. Note that $d_{ij}^{(p)}$ and $d_{ij}^{(J)}$ are fundamentally different, in that $d_{ij}^{(J)}$, or at least its time derivative, is perfectly deterministic (it can be determined from expanding the integral in (1.41)), whereas there is no rigorous theory which can predict the magnitude of the coefficients c_{ijk}, and hence the magnitude of the $d_{ij}^{(p)}$.

Note also that, in isotropic turbulence, the Kármán–Howarth equation tells us that the conditions under which the long-range triple correlations are weak are usually the same as those in which the long-range double correlations are small. However, this is not, in general, true in MHD turbulence. That is, we have $\langle u_i u_j' \rangle_\infty \sim [d_{ij}^{(p)} + d_{ij}^{(J)}] r^{-5} = d_{ij} r^{-5}$, where the time derivatives of the $d_{ij}^{(p)}$ are determined partly by the unknown c_{ijk}. So the conditions under which the $\langle u_i u_j' \rangle_\infty$ vanish are not, in general, the same as those in which the $\langle u_i u_j u_k' \rangle_\infty$ vanish. Indeed, if we want all the coefficients d_{ij} to be zero, and hence $\langle u_i u_j' \rangle_\infty < O(r^{-5})$, we require that certain of the c_{ijk} are 'non-zero'. Clearly we must exercise caution in our handing of the long-range effects in MHD turbulence.

The potentially slow fall-off of $\langle u_i u_j' \rangle_\infty$ and $\langle u_i u_j u_k' \rangle_\infty$ has three important consequences. First, it calls into question the convergence of integrals of type $I_{//}$. In fact, it turns out that, if $\langle u_i u_j' \rangle_\infty \sim r^{-5}$, then this integral is conditionally convergent, and converges if the integrand is evaluated over a large sphere whose radius is allowed to recede to infinity (Batchelor & Proudman 1956; Davidson 1997). However, the fact that $I_{//}$ is only conditionally convergent means that the spectral tensor, $\Phi_{ij}(\boldsymbol{k})$, which is the transform of $\langle u_i u_j' \rangle(\boldsymbol{r})$, is non-analytic at $\boldsymbol{k}=0$, with $\partial^2 \Phi_{ij} / \partial k_n \partial k_m$ being ill-defined at $\boldsymbol{k}=0$ (Batchelor & Proudman 1956). Second, since the slow decline of $\langle u_i u_j u_k' \rangle_\infty \sim r^{-4}$ can lead to a time-dependence of I in isotropic turbulence (see (1.36)), the same triple correlations could, in principle, induce a time-dependence of $I_{//}$ in MHD turbulence. Third, since an $\langle u_i u_j' \rangle_\infty \sim r^{-6}$ fall-off in isotropic turbulence is enough to call into question Landau's derivation of (1.15), it is likely that the potentially slower decline of $\langle u_i u_j' \rangle_\infty \sim r^{-5}$ in MHD turbulence will invalidate the derivation of (1.16) from (1.8).

All in all, it would seem that the predicted conservation laws (1.16) and (1.21) are unlikely to survive if there are significant long-range correlations, and if $u^2 \ell_\perp^4 \ell_{//}$ is not conserved, then the decay laws (1.31)–(1.33) are bound to fail. However, it should be remembered that the magnitude of the coefficients c_{ijk} and $d_{ij}^{(p)}$ remains undetermined by any rigorous theory. Moreover, in conventional isotropic turbulence certain c_{ijk} and d_{ij} are found to be so small that, to a reasonable degree of accuracy, $I =$ constant and $u^2 \sim t^{-10/7}$, at least in fully developed turbulence. Could it be that the appropriate c_{ijk} and d_{ij} are also small in MHD turbulence? If so, there is the possibility that, in fully developed turbulence, $u^2 \ell_\perp^4 \ell_{//}$ is indeed conserved, at least approximately, and that (1.31)–(1.33) survive.

In this paper we explore these issues, but rather than start with the inhomogeneous thought experiment which led to (1.8), and hence to (1.16), we stay within the framework of strictly homogeneous turbulence. We start, in § 2, by exploring the consequences of assuming that certain long-range correlations may be neglected. We shall see that this leads us back to (1.21), and hence to (1.31)–(1.33), but with a number of caveats. Next, we examine the results of numerical simulations performed in very large computational domains. We shall see that, in fully developed turbulence, $u^2 \ell_\perp^4 \ell_{//}$ is more or less constant, and that decay laws (1.31)–(1.33) are surprisingly good approximations. It would seem, therefore, that the situation is similar to that of conventional turbulence: the turbulence evolves so as to minimize certain long-range correlations, so that a naïve theory based on the absence of such long-range interactions fairs unexpectedly well.

2. The behaviour of the Loitsyansky-like integral $I_{//}$

We now re-examine the behaviour of $I_{//}$ in MHD turbulence under the assumption that $\langle u_i u_j' \rangle_\infty$ decays rapidly with separation, i.e. $\langle u_i u_j' \rangle_\infty < O(r^{-5})$. However, rather than start with angular momentum conservation, and the inhomogeneous thought experiment of § 1.2, we shall use the generalized Kármán–Howarth equation, which allows us to stay within the framework of homogeneous turbulence.

Of course, we do not know *a priori* if $\langle u_i u_j' \rangle_\infty$ will fall off faster than r^{-5}, and indeed (1.43) tells us that it is somewhat unlikely. However, our experience with isotropic turbulence suggests that, after an initial transient, the pre-factors which multiply the leading-order far-field terms can be very small (Ishida *et al.* 2006), so small that they may be ignored for many purposes. So it makes sense to at least explore the consequences of ignoring the $\langle u_i u_j' \rangle_\infty \sim O(r^{-5})$ tail, and then compare the resulting predictions with the results of numerical experiments.

2.1. *A real-space analysis of $I_{//}$*

Let us suppose that $\boldsymbol{B}_0$ points in the z-direction. Then we may write $I_{//}$ as

$$I_{//} = -\int \left[r_x^2 \langle u_y u_y' \rangle + r_y^2 \langle u_x u_x' \rangle \right] \mathrm{d}\boldsymbol{r}. \tag{2.1}$$

Note that we have omitted integrals of the form $\int [r_x^2 \langle u_x u_x' \rangle] \mathrm{d}\boldsymbol{r}$ from (2.1). This is justified because the integrand can be expressed as a divergence in $\boldsymbol{r}$, which integrates to zero. That is, $r_x^2 u_x' = (1/3) \nabla_{\boldsymbol{r}} \cdot (r_x^3 \boldsymbol{u}')$ and so, if S is a large spherical surface, we have

$$\int \left[r_x^2 \langle u_x u_x' \rangle \right] \mathrm{d}\boldsymbol{r} = \left\langle u_x(\boldsymbol{r}=0) \int r_x^2 u_x' \mathrm{d}\boldsymbol{r} \right\rangle = \frac{1}{3} \left\langle u_x(\boldsymbol{r}=0) \oint_S r_x^3 \boldsymbol{u}' \cdot \mathrm{d}\boldsymbol{S} \right\rangle, \tag{2.2}$$

which vanishes if $\boldsymbol{u}$ and $\boldsymbol{u}'$ decorrelate faster than r^{-5}.

Now the generalized Kármán–Howarth equation is, from (1.40),

$$\frac{\partial}{\partial t} \langle u_i u_j' \rangle = \nabla \cdot \left[\langle u u u' \rangle , \langle p u' \rangle \right] - \frac{2}{\tau} \nabla^{-2} \frac{\partial^2}{\partial r_z^2} \langle u_i u_j' \rangle, \tag{2.3}$$

and we know from Batchelor (1953) that, in the absence of the Lorentz force, integrals of the type

$$I = -\int \boldsymbol{r}^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r}, \quad \text{and} \quad I_{//} = -\int \boldsymbol{r}_\perp^2 \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle \, \mathrm{d}\boldsymbol{r},$$

are invariants when the triple correlations decay faster than $O(r^{-4})$. Although Batchelor's analysis is performed in Fourier space, its results can be confirmed in real space by integrating the second moments of (2.3) over $\boldsymbol{r}$ and rewriting the terms involving $\nabla \cdot [\langle uuu' \rangle, \langle pu' \rangle]$ as divergences in $\boldsymbol{r}$, which disappear upon integration. However, when $\langle u_i u_j u_k' \rangle_\infty \sim r^{-4}$, these divergences do not vanish on integration, and so the integrals I and $I_{//}$ become time-dependent.

Since the influence of the triple correlations on I and $I_{//}$ are understood, at least qualitatively, we now concentrate on the role played by the Lorentz force in (2.3). Let the symbol [NL] indicate any term which involves the triple correlations, $\langle uuu' \rangle$, or the pressure–velocity correlations, $\langle pu' \rangle$, i.e. any term which arises from the nonlinear contributions to the Navier–Stokes equation. Then, from (2.3) we have

$$\frac{\partial}{\partial t} \nabla^2 \langle u_i u_j' \rangle = [\mathrm{NL}] - \frac{2}{\tau} \frac{\partial^2}{\partial r_z^2} \langle u_i u_j' \rangle. \tag{2.4}$$

Multiplying by $r_x^2 r_y^2$ then yields

$$2 r_\perp^2 \frac{\partial}{\partial t} \langle u_i u_j' \rangle = [\mathrm{NL}] + \nabla \cdot \left[\langle u u' \rangle \right], \tag{2.5}$$

where $\nabla \cdot [\langle uu' \rangle]$ indicates the divergence of terms involving $\langle u_i u_j' \rangle$. Integrating over all $\boldsymbol{r}$, and assuming that $\langle u_i u_j' \rangle_\infty$ decays faster than $|\boldsymbol{r}|^{-5}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[-\int \boldsymbol{r}_\perp^2 \langle u_i u_j' \rangle \, \mathrm{d}\boldsymbol{r} \right] = \int [\mathrm{NL}] \, \mathrm{d}\boldsymbol{r}, \tag{2.6}$$

which confirms that the Lorentz force has no direct influence on $I_{//}$. Moreover, we know from conventional homogeneous turbulence that the integral on the right-hand side of (2.6) is zero when certain triple correlations fall faster than $O(r^{-4})$, and so (2.6) yields

$$I_{//} = \text{constant} \quad \text{if } \langle u_i u_j u'_k \rangle_\infty < O(r^{-4}). \tag{2.7}$$

The behaviour of $I_{//}$ in (2.7) is as expected: if certain long-range triple correlations are weak, then $I_{//}$ = constant, whereas the long-range triple correlations envisaged by Batchelor & Proudman (1956) will make $I_{//}$ time-dependent. In short, $I_{//}$ behaves just like I in conventional isotropic turbulence.

2.2. *A Fourier space analysis of* $I_{//}$

Similar conclusions can be reached by examining the equations in Fourier space. Since the Fourier space interpretation will prove useful when we come to examine the DNS, we briefly summarize the arguments here. Introducing the spectral tensor

$$\Phi_{ij}(\boldsymbol{k}) = (2\pi)^{-3} \int \langle u_i u'_j \rangle (\boldsymbol{r}) \exp(-j\boldsymbol{k}\cdot\boldsymbol{r})\, d\boldsymbol{r}, \tag{2.8}$$

and expanding the exponential in a power series in $\boldsymbol{k}\cdot\boldsymbol{r}$, we find

$$(2\pi)^3 \Phi_{ij}(\boldsymbol{k}) = \int \langle u_i u'_j \rangle\, d\boldsymbol{r} - jk_p \int r_p \langle u_i u'_j \rangle\, d\boldsymbol{r} - \frac{1}{2} \int (\boldsymbol{k}\cdot\boldsymbol{r})^2 \langle u_i u'_j \rangle\, d\boldsymbol{r} + O(k^3). \tag{2.9}$$

The first term on the right-hand side may be converted into a surface integral, which vanishes under the assumption that $\langle u_i u'_j \rangle_\infty$ decays faster than r^{-3}. The second term also goes to zero if we restrict ourselves to $i = j$, since homogeneity then requires $\langle u_i u'_j \rangle(\boldsymbol{r}) = \langle u_j u'_i \rangle(-\boldsymbol{r}) = \langle u_i u'_j \rangle(-\boldsymbol{r})$, which tells us that the integrand is odd in $\boldsymbol{r}$. (Actually, the second integral is zero under less restrictive conditions, but that need not concern us here.) Expanding $(\boldsymbol{k}\cdot\boldsymbol{r})^2$ in the third integral we find, after a little algebra,

$$32\pi^3 \Phi_\perp(\boldsymbol{k} \to 0) = -k_\perp^2 \int \boldsymbol{r}_\perp^2 \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle\, d\boldsymbol{r} - 2k_z^2 \int r_z^2 \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle\, d\boldsymbol{r} + O(k^3) \tag{2.10}$$

or, in terms of $I_{//}$,

$$32\pi^3 \Phi_\perp(\boldsymbol{k} \to 0) = I_{//} k_\perp^2 + I_\perp k_z^2, \tag{2.11}$$

where

$$I_\perp = -2 \int_{V_\infty} r_z^2 \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle\, d\boldsymbol{r}. \tag{2.12}$$

In spectral space, then, we have the kinematic relationship

$$I_{//} = \lim_{k_\perp \to 0} \frac{32\pi^3 \Phi_\perp(k_z = 0, k_\perp)}{k_\perp^2}. \tag{2.13}$$

So far we have not had to invoke the restriction that $\langle u_i u'_j \rangle_\infty < O(r^{-5})$. However, this restriction is in fact implicit in (2.13) since, as we have seen, $\langle u_i u'_j \rangle_\infty \sim O(r^{-5})$ implies that $I_{//}$ is only conditionally convergent, implying that its value may depend on the way in which the boundary of the domain of integration recedes to infinity. Turning now to dynamics, the transform of (2.3) is

$$\frac{\partial \Phi_{ij}}{\partial t} = [\text{NL}] - \frac{2}{\tau} \frac{k_z^2}{k^2} \Phi_{ij}, \tag{2.14}$$

and if the inertial terms on the right-hand side can be ignored at $O(k^2)$, this yields

$$\Phi_{ij}(\boldsymbol{k}, t) = \Phi_{ij}(t = 0) \exp\left[-\frac{k_z^2}{k^2}\frac{2t}{\tau}\right] + O(k^3). \tag{2.15}$$

(In the absence of long-range triple correlations, $\langle u_i u_j u_k' \rangle_\infty = c_{ijk} r^{-4}$, the nonlinear terms in (2.14) are of order k^3; Batchelor 1953.) Note that (2.13) probably requires that $\langle u_i u_j' \rangle_\infty$decays as $\langle u_i u_j' \rangle_\infty < O(r^{-5})$, but makes no assumption about the triple correlations, whereas (2.15) assumes that the triple correlations decay rapidly with separation, i.e $\langle u_i u_j u_k' \rangle_\infty < O(r^{-4})$, but makes no assumptions about the double correlations.

One advantage of this spectral approach is that it brings out the relationship between $I_{//}$, and $\Phi_\perp(\boldsymbol{k} \to 0)$. This will prove useful when examining the results of the numerical simulations. To this end it is convenient to introduce the quantity $J_{//}$, defined by

$$J_{//} = \lim_{k_\perp \to 0} \frac{32\pi^3 \Phi_\perp(k_z = 0, k_\perp)}{k_\perp^2}. \tag{2.16}$$

Of course, provided $\langle u_i u_j' \rangle_\infty < O(r^{-5})$, $J_{//}$ satisfies $J_{//} = I_{//}$, and we might expect $J_{//} \approx I_{//}$ if certain d_{ij} in (1.44) are small. Moreover (2.15) yields

$$J_{//} = \text{constant}, \tag{2.17}$$

provided that $\langle u_i u_j u_k' \rangle_\infty < O(r^{-4})$, and we expect $J_{//} \approx$ constant if certain c_{ijk} are weak.

Now$J_{//} \sim u^2 \ell_\perp^4 \ell_{//}$, so that (2.17) plus self-similarity of the large scales allows us to recover $u^2 \ell_\perp^4 \ell_{//} =$ constant. Thus (1.21), and hence the decay laws (1.31)–(1.33) can be obtained from either $I_{//} =$ constant or $J_{//} =$ constant. We shall see that $J_{//}$ is much easier to evaluate from the simulations than $I_{//}$, because the integral $I_{//}$ converges rather slowly and we quickly hit the finite size of the periodic domain. Thus we shall place more emphasis on $J_{//}$ than on $I_{//}$ in our interpretation of the numerical results.

3. The numerical evidence

We now turn to the evidence of the numerical simulations. The questions we seek to answer are as follows.

(i) Are $I_{//}$ and $J_{//}$ approximately constant in fully developed turbulence, as predicted by (1.16) and (2.17)?

(ii) Are the large scales self-similar, so that the conservation of $I_{//}$ and $J_{//}$ translates to $u^2 \ell_\perp^4 \ell_{//} =$ constant?

(iii) If $u^2 \ell_\perp^4 \ell_{//}$ is conserved in fully developed turbulence, are the other assumptions in the decay model of Davidson (2001, 2004) reasonable approximations, i.e. are α, β and C really constant once the turbulence matures?

(iv) If the answers to (i)–(iii) are yes, does MHD turbulence decay in accordance with (1.31)–(1.33)?

Note that by ‘fully developed turbulence’ we mean turbulence that has largely forgotten its initial conditions, which in any event are somewhat unphysical (i.e. random phases of the Fourier modes).

The simulations reported here employ the spectral code described in Ishida *et al.* (2006). The boundary conditions are periodic and the random initial conditions were chosen from a Gaussian ensemble with $\langle \boldsymbol{u}^2 \rangle_{t=0} = 1$. The prescribed initial energy spectrum is isotropic and has the form $E \sim k^4 \exp[-2(k/k_p)^2]$, where k_p is

Run	M	k_p	$Re(t=0)$	N_0	t_{max}/T	k_{max}/k_p
1	1024^3	40	362	7/60	300	12
2	1024^3	60	181	7/60	300	8
3	$512^2 \times 1024$	40	181	7/30	320	6
4	2048^3	80	181	7/30	256	12
5	1024^3	60	90	7/60	300	8

Table 1. Parameters in the simulations.

the wavenumber at which $E(k, t=0)$ is a maximum. Note that, with this choice of $E(t=0)$, ℓ is related to k_p at $t=0$ by $\ell = \sqrt{2\pi}/k_p$, where ℓ is defined in the usual way as the integral of the longitudinal correlation function. The phase-shift method was used for de-aliasing, in which the maximum wavenumber, k_{max}, of the retained Fourier modes is about $2^{1/2}N/3$. The minimum wavenumber is $k_{min}=1$.

The details of the simulation are listed in table 1. Here M is the number of Fourier modes in each direction, $Re = u\ell_\perp/\nu$, N_0 is the initial value of the interaction parameter, and time is normalized by the initial eddy turn-over time, defined as $T = 1/\langle \boldsymbol{u}^2\rangle_0^{1/2} k_p = 1/k_p$. Note that, if L_{BOX} is the domain size, then $k_p L_{BOX} = 2\pi k_p$, which implies that $L_{BOX}/\ell = (2\pi)^{1/2} k_p$ at $t=0$. This corresponds to $L_{BOX}/\ell = 100$–200 in our simulations.

In the hydrodynamic decay simulations of Ishida *et al.* (2006) it was found that those runs whose initial value of Re exceeded 180 exhibited no dependence of $u^2(t)$ and $\ell(t)$ on Re, whereas those in which $Re(t=0)$ was less than 100 did exhibit some influence of Re on the behaviour of the large scales. Hence we might expect that runs 1–4 will be indicative of high-Re turbulence, but that there should be some influence of viscosity on the decay exponent for $u^2(t)$ in run 5. We shall therefore focus for the most part on runs 1–4.

Let us start with the issue of whether or not (2.13), which requires $\langle u_i u'_j\rangle_\infty < O(r^{-5})$, provides reliable estimates of $I_{//}$. Of course, in the light of (1.43), it is virtually certain that there exists terms of order $\langle u_i u'_j\rangle_\infty \sim O(r^{-5})$. However, the question before us is whether these are strong enough to cause a significant difference between $J_{//}$ and $I_{//}$. Figure 1(a) shows a comparison between $I_{//}$ and $J_{//}$ for run 4 at $t/T=112$, where $I_{//}$ was evaluated from the integral of $\langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp\rangle$ in accordance with definition (1.16), while $J_{//}$ was calculated using definition (2.16) and a best fit to the spectral data in the range $0.07 < k_\perp/k_p < 0.1$. Note that $\langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp\rangle$ was obtained by Fourier-transforming $\Phi_\perp(\boldsymbol{k})$, which is shown in figure 1(b). The integral $I_{//}$ was evaluated in three different ways; over a sphere of increasing radius, r, over a cylinder of volume $2\pi r^3$, and over a cube of side $2r$. All three are plotted in figure 1(a) as a function of $k_p r$. It can be seen that all three methods of evaluating $I_{//}$ appear to asymptote to $J_{//}$ for $k_p r < 25$ (i.e. $r/\ell < 10.0$), but the comparison is not conclusive because the integral converges slowly and the periodicity of the domain starts to play a role for $k_p r > 25$, causing oscillations in the integrals. That is to say, for $k_p r > 25$, which corresponds to $k/k_p < 0.1$, there are a limited number of Fourier modes contributing to $\Phi_\perp(\boldsymbol{k})$, and hence some statistical scatter in $\Phi_\perp$, as seen in figure 1(b). Thus, due to the lack of a good statistical sample, the transform of $\Phi_\perp$, i.e. $\langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp\rangle$, is subject to scatter for $r/\ell > 10.0$, and it is this which, on integration, produces the oscillations in $I_{//}$ for $k_p r > 25$. It is important to note, however, that these oscillations have nothing to do with the convergence properties of $I_{//}$, but rather arise from the finite size of L_{BOX}, i.e. from periodicity.

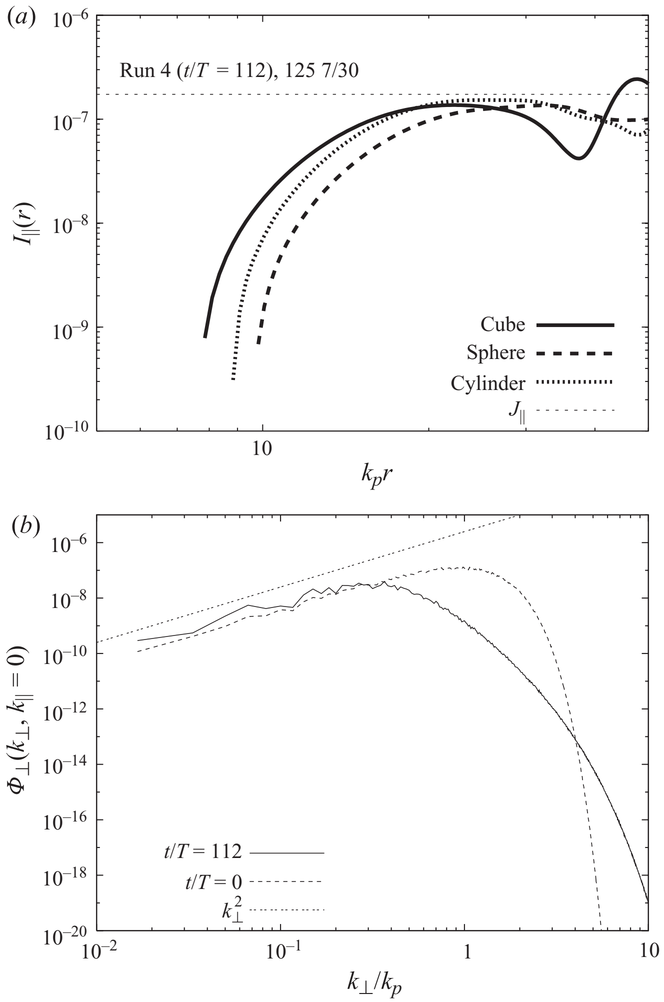

FIGURE 1. (*a*) A comparison of $I_{//}$ and $J_{//}$ at $t/T = 112$ for run 4. (*b*) The spectrum $\Phi_{\perp}(\boldsymbol{k})$ from which $\langle \boldsymbol{u}_{\perp} \cdot \boldsymbol{u}'_{\perp} \rangle$ was obtained.

The apparent convergence of $I_{//}$ to $J_{//}$ for $k_p r < 25$ suggests that the properties of $I_{//}$ are well behaved, despite the far-field terms, $\langle u_i u'_j \rangle_\infty \sim O(r^{-5})$. It should be noted, however, that this is not true of other integrals of similar form. For example, the convergence properties of $I_{\perp}$ show a strong dependence on the shape of the domain of integration and so are clearly influenced by the $\langle u_i u'_j \rangle_\infty \sim O(r^{-5})$ terms. In any event, since the invariance of either $J_{//}$ or $I_{//}$ is enough to justify $u^2 \ell_{\perp}^4 \ell_{//} = \text{constant}$ (assuming self-similarity applies), and $J_{//}$ is much easier to evaluate unambiguously than $I_{//}$, we shall focus on $J_{//}$ from now on.

Let us now consider the behaviour of $J_{//}$ as a function of time. Figure 2 shows $J_{//}(t)/J_{//}(0)$ and $\mathrm{d}J_{//}/\mathrm{d}t$ for runs 1–4, where $J_{//}$ is estimated from $\Phi_{\perp}(\boldsymbol{k} \to 0)$ using (2.16) and spectral data in the range $0.07 < k_{\perp}/k_p < 0.1$. In runs 1–3 $J_{//}$ reaches a constant value after an initial transient (i.e. for $t > 70T$), whereas run 4 exhibits a small but finite growth of $J_{//}$ at large times. However, figure 2(*b*) shows that this growth in run 4 is small, and so we conclude that $J_{//}$ is indeed approximately constant

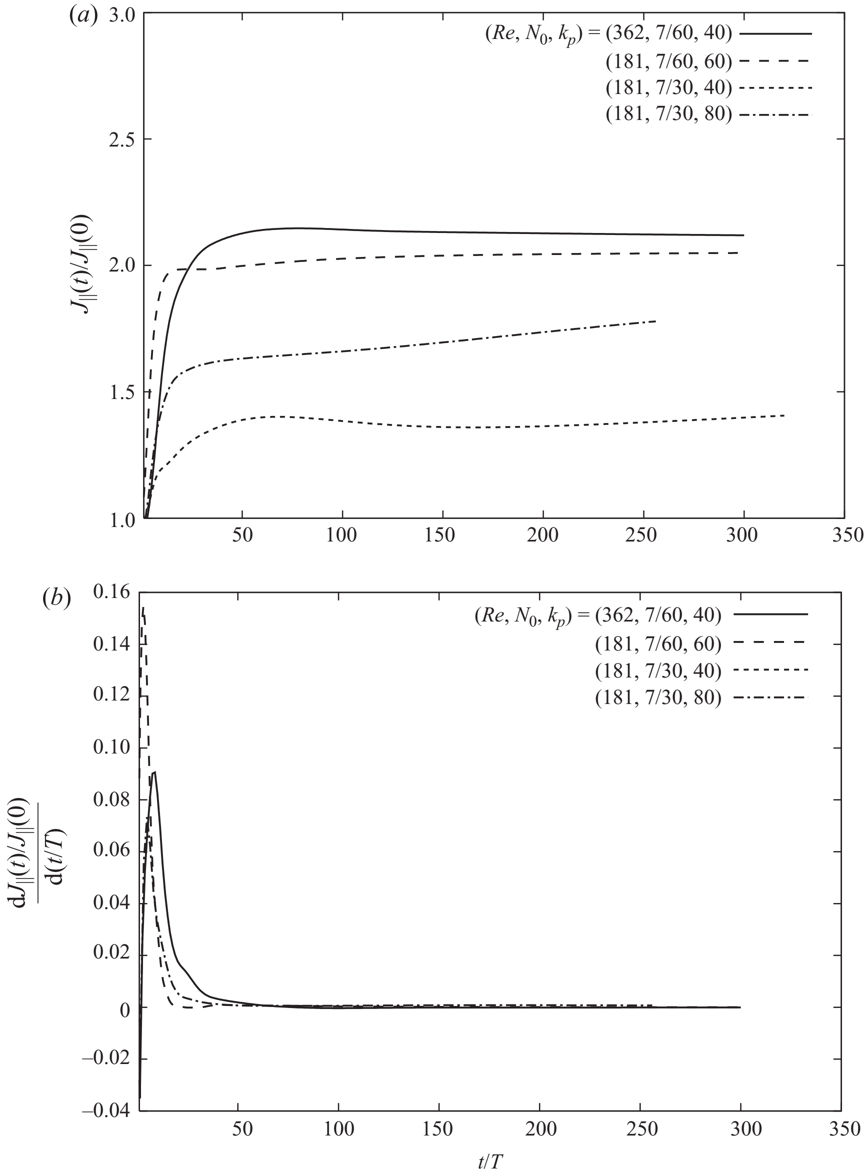

FIGURE 2. Time-dependence of $J_{//}$ for runs 1–4: (*a*) $J_{//}(t)/J_{//}(0)$, (*b*) $(\mathrm{d}J_{//}/\mathrm{d}t)(T/J_{//}(0))$.

after the transient, consistent with (2.17) and with the behaviour of I in isotropic turbulence.

Given that $J_{//}$ is approximately constant for $t > 70T$, we now ask if $u^2\ell_\perp^4\ell_{//} = \text{constant}$ in the same time interval, i.e. how good an approximation is self-similarity of the large scales. The evolution of $u^2\ell_\perp^4\ell_{//}$ is shown in figure 3 for the period $1 < t/T < 300$, both in a linear plot and in log–log form. It can be seen that, despite $J_{//}$ being more or less constant for $t > 70T$, $u^2\ell_\perp^4\ell_{//}$ rises slowly during the same time period, showing that the large scales are not perfectly self-similar. Nevertheless, we shall see shortly that this slow rise in $u^2\ell_\perp^4\ell_{//}$ is weak enough to leave (1.31)–(1.33) good approximations to fully developed turbulence.

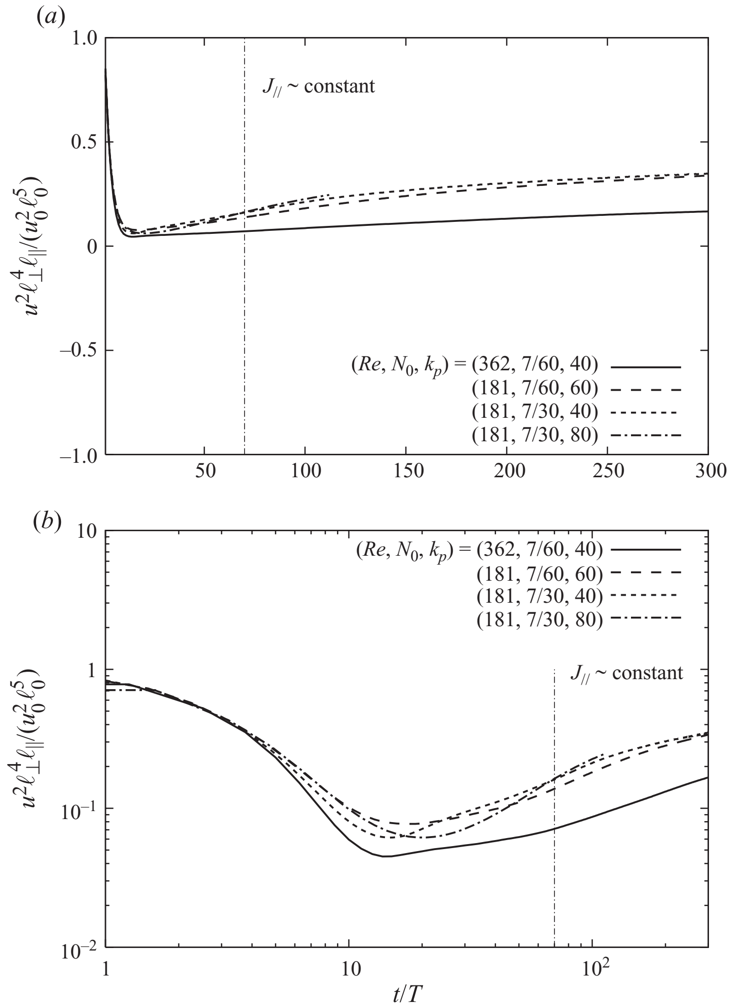

FIGURE 3. The time-dependence of $u^2\ell_\perp^4\ell_{//}$ in runs 1–4 for $1 < t/T < 300$: (*a*) linear plot, (*b*) log–log plot.

The variations of $\ell_{//}/L_{BOX}$ and $Re = u\ell_\perp/\nu$ with t/T are shown in figure 4. It can be seen that $\ell_{//}/L_{BOX} < 0.08$ in all cases, so that we might hope that the unphysical boundary conditions (periodicity) do not unduly influence the simulations. On the other hand, the values of Re become rather modest at large times, so there is the possibility of finite-Reynolds-number effects. However, we shall see shortly that this is probably not the case for runs 1–4.

Given that, after a transient, $J_{//}$ is indeed approximately constant, it seems appropriate to investigate the decay model outlined in § 1.3. The additional model assumptions, over and above $J_{//} = \text{constant}$, are that α, β and C are all constant in fully developed turbulence, with $C = 1$. Figure 5 shows the variation of α, β and C for runs 1–4. It is reassuring that, after the usual transient, α and β are indeed approximately constant and of order unity. The coefficient C, defined by (1.35), is also reasonably constant at large times, though its value depends

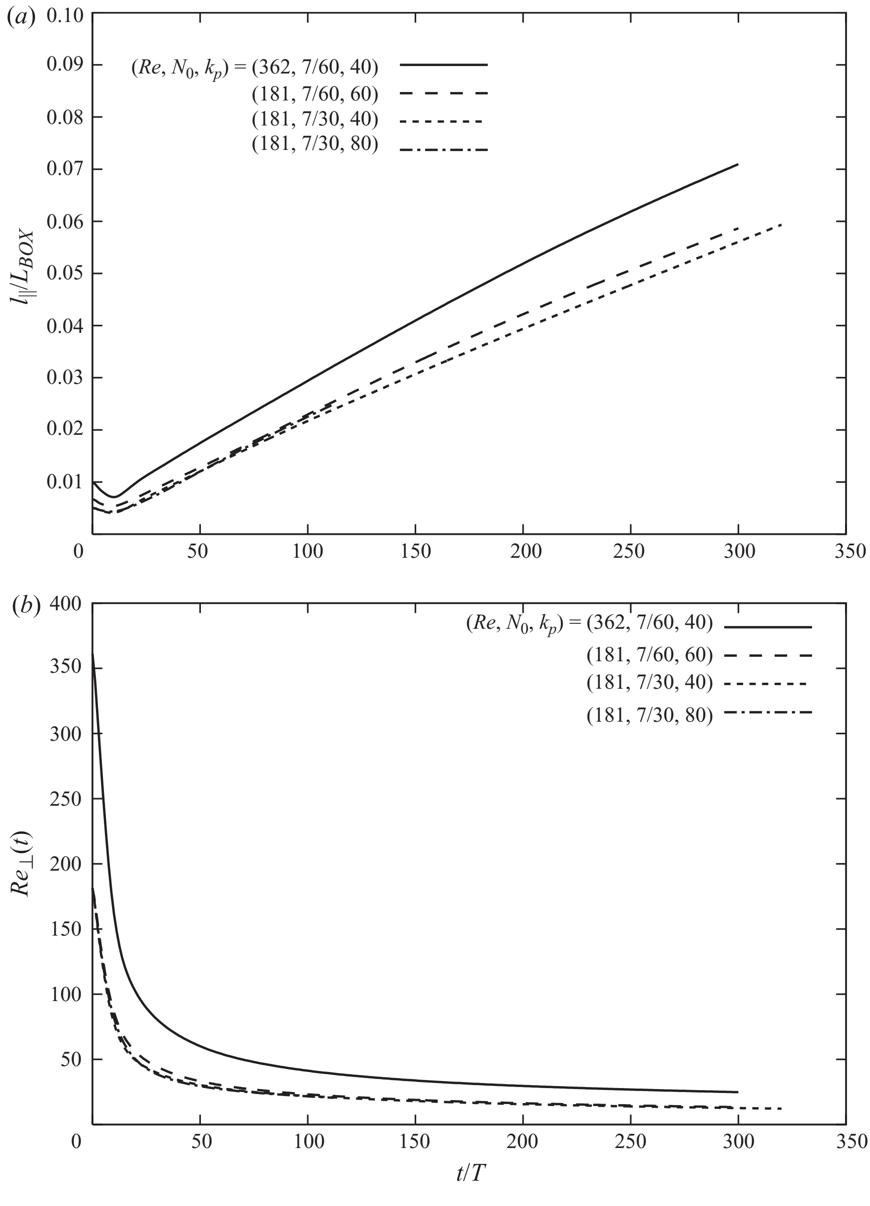

FIGURE 4. Time-dependence of (a) $\ell_{//}/L_{BOX}$, (b) $Re = u\ell_{\perp}/\nu$, for runs 1–4.

somewhat on N_0 and can differ significantly from unity. Thus we conclude that the weakest ingredient in the model of Davidson (2001, 2004) is the heuristic equation (1.30).

Also shown in figure 5 are the values of $7\alpha/15\beta N_0$. This is included because, for the particular value of $N_0 = 7\alpha/15\beta$, predictions (1.31)–(1.33) reduce to the simple power laws $u^2 \sim u_0^2\hat{t}^{-11/7}$, $\ell_{//} \sim \ell_0\hat{t}^{5/7}$ and $\ell_{\perp} \sim \ell_0\hat{t}^{3/14}$. The values of $7\alpha/15\beta N_0$ are significantly different from unity, typically around 3–4. However, for $t/T > 150$, (1.31) suggests that this departure from $7\alpha/15\beta N_0 = 1$ should have only a modest effect on

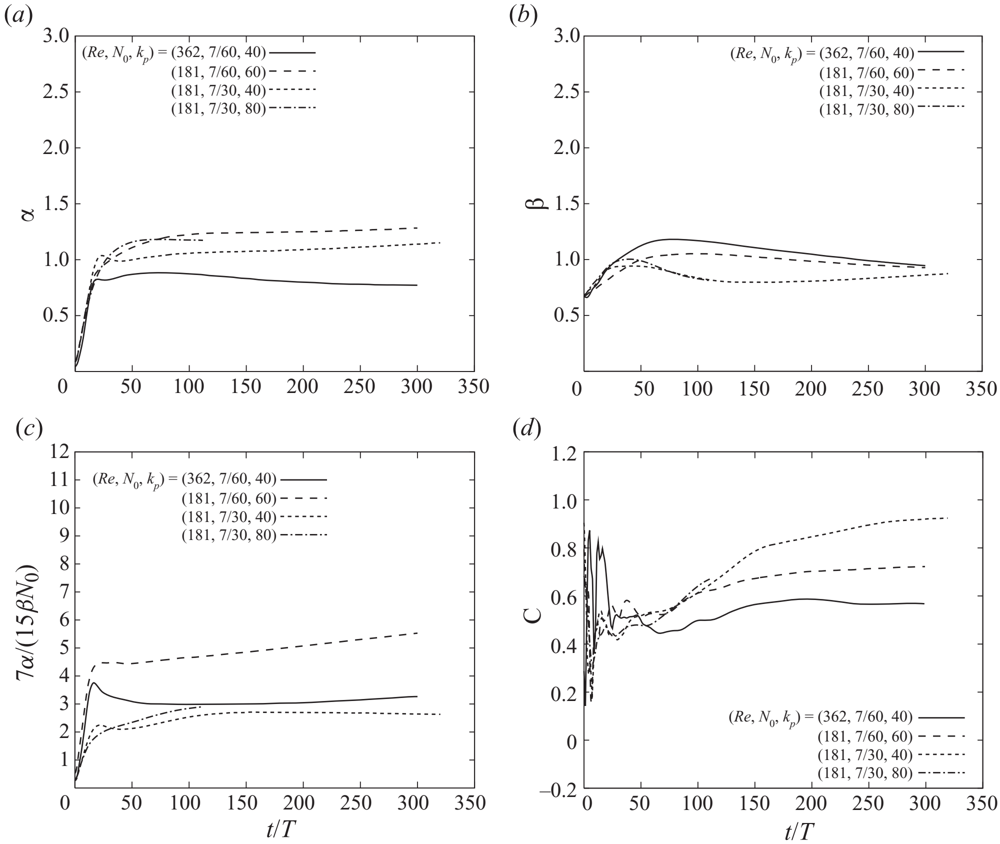

FIGURE 5. Time-dependence of α, β, $7\alpha/15\beta N_0$ and C for runs 1–4.

the predicted behaviour of $u^2(t)$, and so we might still expect to see the power law $u^2 \sim u_0^2 \hat{t}^{-11/7}$ in runs 1–4.

Figure 6(*a*) shows the local value of the computed exponent $m(t)$ in the power law estimate $u^2 \sim u_0^2 (t/T)^{-m}$ for runs 1–4. For large t/T the exponent does indeed come close to the predicted value of 11/7. Figure 6(*b*) shows the same data for runs 2 and 5. In run 5, where $Re(t=0)$ is somewhat low, the exponent m is significantly larger than 11/7. This is consistent with the results of conventional isotropic turbulence, where initial values of Re similar to that in run 5 produced energy decay rates which were different to, and higher than, those for larger Re (Ishida *et al.* 2006).

Figure 7 shows the compensated plots of $u^2/\hat{t}^{-11/7}$, $\ell_{//}/\hat{t}^{5/7}$ and $\ell_{\perp}/\hat{t}^{3/14}$ versus t/T in both linear and log–log form. The power-law dependencies of (1.34) should correspond to plateaux in each of these figures and this is indeed observed for the energy decay $u^2(t)$, although this is less convincing in the case of $\ell_{//}$ and $\ell_{\perp}$. However, the fact that the power laws (1.34) are not clear-cut for $\ell_{//}$ and $\ell_{\perp}$ is associated, in part, with the fact that $7\alpha/15\beta N_0 \neq 1$ in these runs, as we now show.

Figure 8 shows u^2/u_0^2, $\ell_{//}/\ell_0$, and $\ell_{\perp}/\ell_0$, all normalized by the corresponding model prediction in (1.31)–(1.33). In each case the compensated plots show clear plateaux

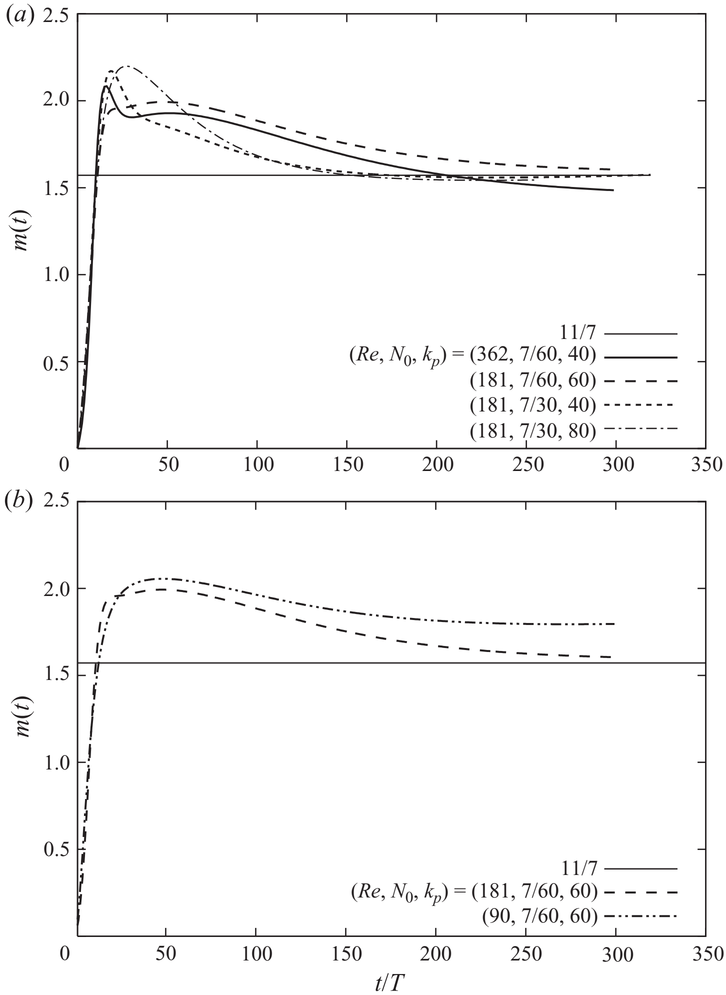

FIGURE 6. Time-dependence of the local value of the exponent $m(t)$ in the power law $u^2 \sim u_0^2(t/T)^{-m}$. (*a*) Runs 1–4. (*b*) Runs 2 and 5. Run 5 is anomalous as *Re* is too low.

at large times (i.e. $t > 70T$), giving strong support to predictions (1.31)–(1.33). The fact that the plateaux are not equal to unity in figure 8(*a*) is because the energy lost during the initial transient is not governed by (1.31).

All in all, it would seem that the model assumptions and predictions have held up well, with the possible exception of the heuristic equation (1.30). The final figure, figure 9, shows snapshots of the flow field in runs 1 and 2 at $t/T = 300$. The vorticity and velocity fields are visualized in each case at the threshold $|\omega| = \langle|\omega|\rangle + 3\sigma$ and $|\boldsymbol{v}| = \langle|\boldsymbol{v}|\rangle + 2\sigma'$, where σ and σ' are the standard deviation of $|\omega|$ and $|\boldsymbol{v}|$, respectively. The anisotropic streaky structure of the eddies is quite evident, particularly in the constant energy surfaces. Moreover, the final value of $\ell_{//}/L_{BOX} \sim 0.07$ is consistent with a qualitative inspection of these figures.

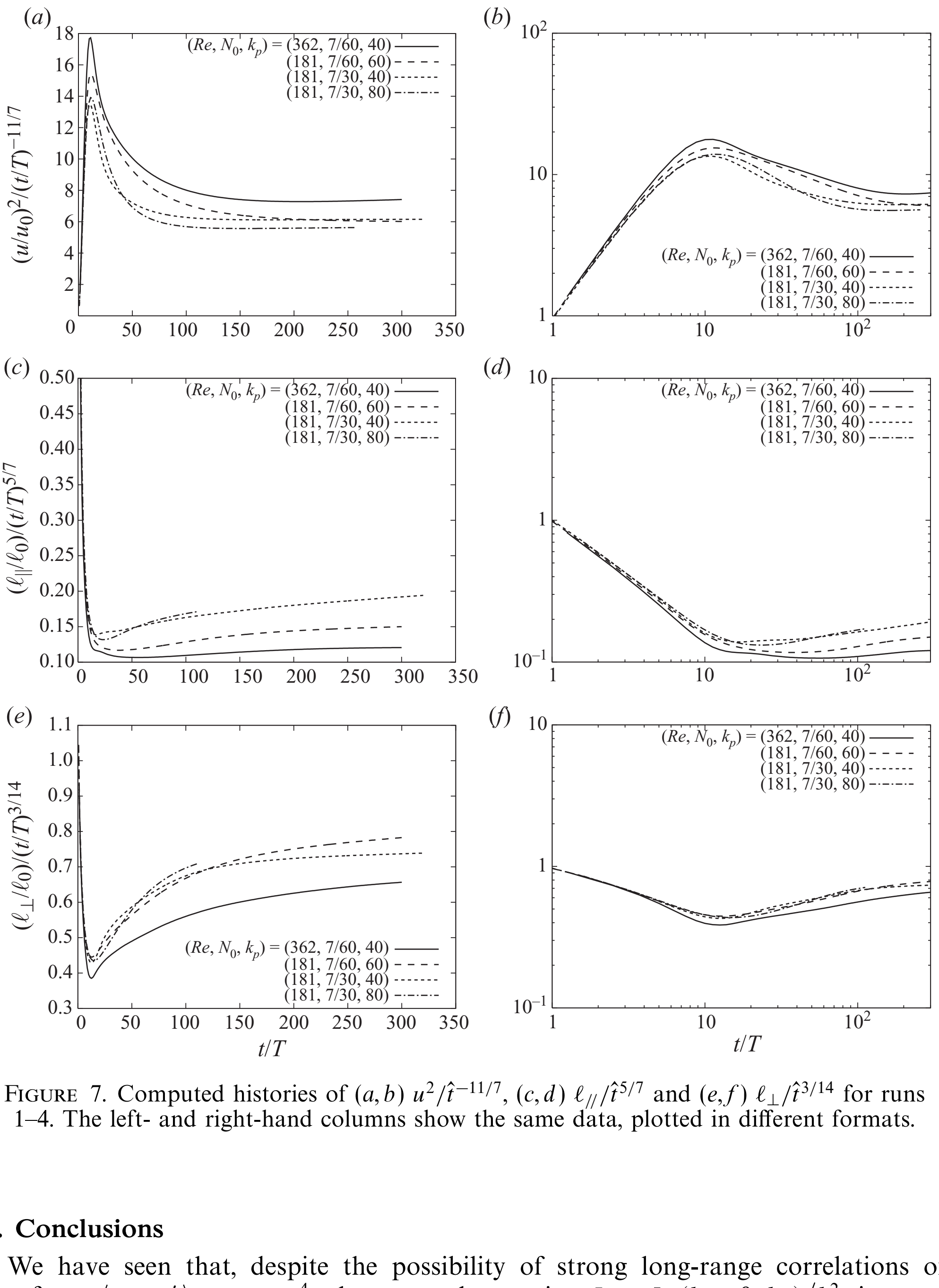

FIGURE 7. Computed histories of (a,b) $u^2/\hat{t}^{-11/7}$, (c,d) $\ell_{/\!/}/\hat{t}^{5/7}$ and (e,f) $\ell_\perp/\hat{t}^{3/14}$ for runs 1–4. The left- and right-hand columns show the same data, plotted in different formats.

4. Conclusions

We have seen that, despite the possibility of strong long-range correlations of the form $\langle u_i u_j u'_k \rangle_\infty = c_{ijk} r^{-4}$, the spectral quantity $J_{/\!/} \sim \Phi_\perp(k_z = 0, k_\perp)\big/k_\perp^2$ is more or less conserved in fully developed MHD turbulence. This, in turn, suggests that the Loitsyansky-like integral $I_{/\!/} = -\int r_\perp^2 \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle \,\mathrm{d}\boldsymbol{r}$ is an invariant of the decay, and also that $u_\perp^2 \ell_\perp^4 \ell_{/\!/} \approx$ constant in the fully developed state, though self-similarity of the large scales is far from perfect. This lends support to the decay model of Davidson (2001, 2004) which, subject to the additional assumptions that α and β are

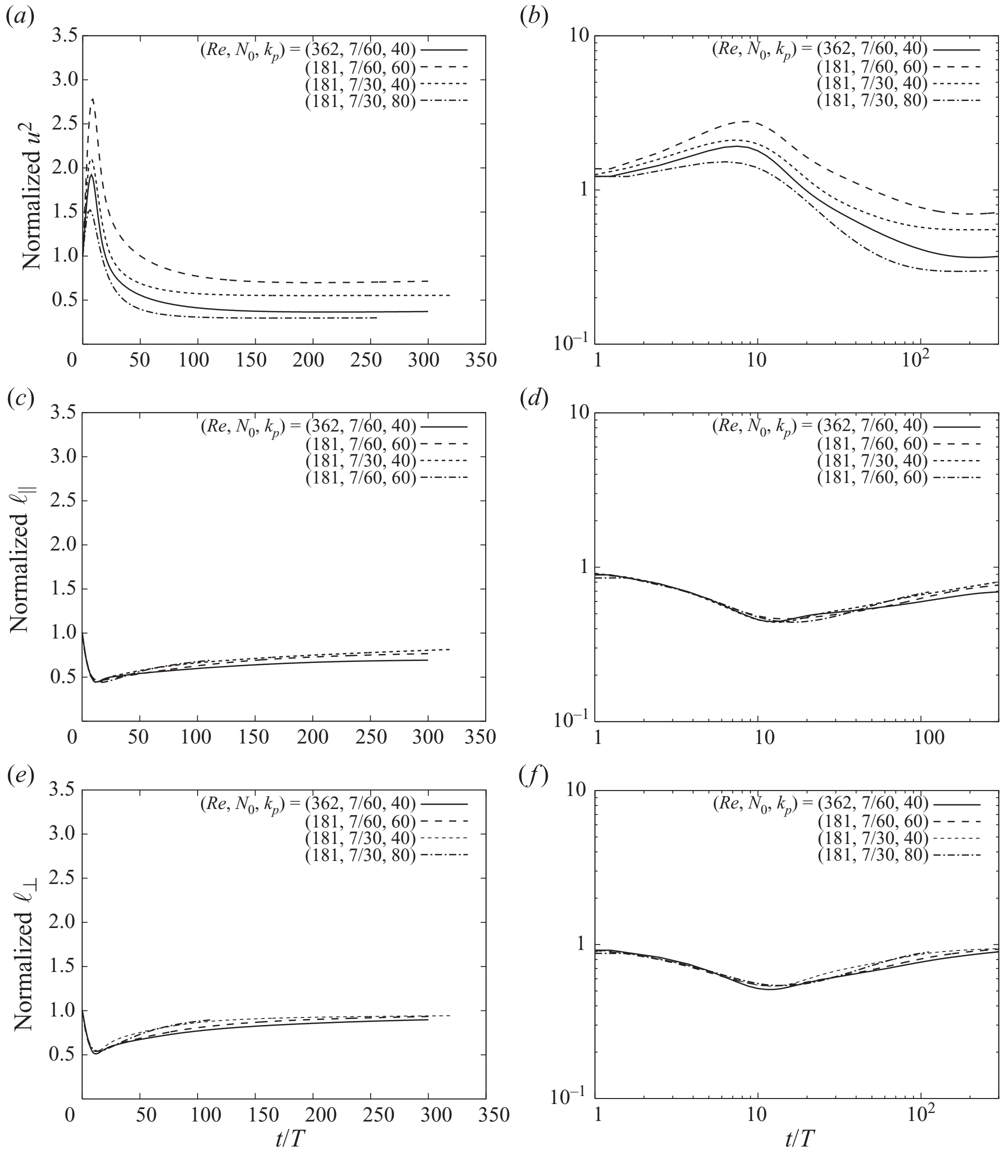

FIGURE 8. Computed histories of u^2/u_0^2, $\ell_{//}/\ell_0$, and $\ell_\perp/\ell_0$, normalized by (1.31)–(1.33), in both linear and log–log plots.

approximately constant, and that the interpolation formula (1.30) is a reasonable approximation, predicts the decay laws (1.31)–(1.33) for $u^2(t\,;N_0)$, $\ell_\perp(t\,;N_0)$ and $\ell_{//}(t\,;N_0)$. The results of the numerical simulations are reasonably consistent with these predictions.

There are many possible extensions to this study, the most obvious being to consider high-R_m turbulence and MHD turbulence in the presence of a bulk rotation or stratification. As noted in Davidson (2004, 2009), all of these systems

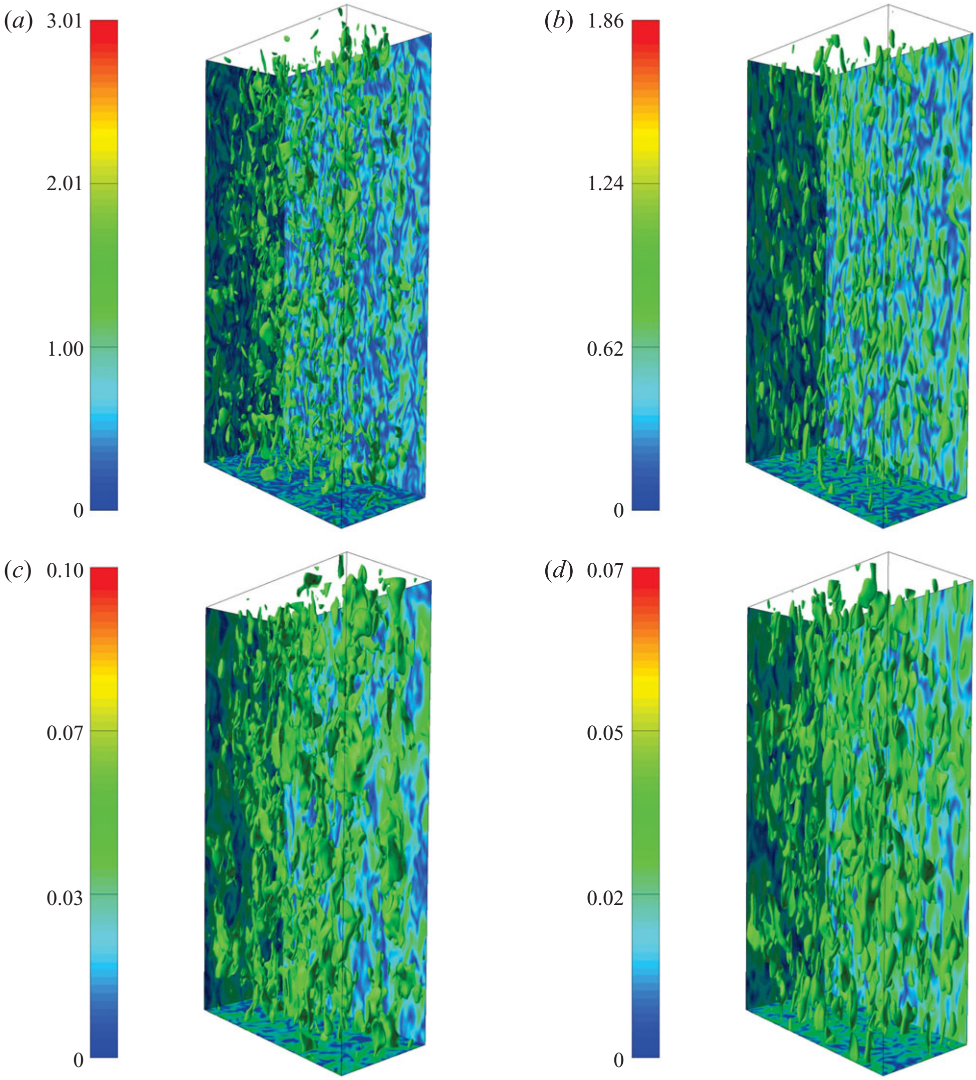

FIGURE 9. Intense vorticity regions at $t/T = 300$ in runs 1 (a) and 2 (b), and intense velocity regions in runs 1 (c) and 2 (d). The thresholds are $|\omega| = \langle|\omega|\rangle + 3\sigma$ and $|\boldsymbol{v}| = \langle|\boldsymbol{v}|\rangle + 2\sigma'$, where σ and $\sigma'$are the standard deviation of $|\omega|$ and $|\boldsymbol{v}|$, respectively. Only the subcubes $512 \times 256 \times 1024$ are visualized.

possess the invariant given in (1.16), subject, of course, to the caveats outlined in § 1.

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